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General formulation and analysis of hyperbolic heat conduction in composite media

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Abstract—Some recent experimental results show the existence of reflections of thermal waves at the interface of dissimilar materials in superfluid helium. In light of these results, a theoretical investigation of thermal waves in composite is provided to give a theoretical foundation to the observed phenomenon. A general one-dimensional temperature and heat flux formulation for hyperbolic heat conduction in a composite medium is presented. Also, the general solution, based on the flux formulation, is developed for the standard three orthogonal coordinate systems. Unlike classical parabolic heat conduction, heat conduction based on the modified Fourier's law produces non-separable field equations for both the temperature and flux and therefore standard analytical techniques cannot be applied in these situations. In order to alleviate this difficulty, a generalized finite integral transform technique is proposed in the flux domain and a general solution is developed for the standard three orthogonal coordinate systems. The general solution is applied to the case of a two-region slab with a pulsed volumetric source and insulated exterior surfaces which displays the unusual and controversial nature associated with heat conduction based on the modified Fourier's law in composite regions.

INTRODUCTION

THE CONSTITUTIVE relation which appears in classical heat conduction is Fourier's law

$$q = -k\frac{\partial T}{\partial x}. \tag{1}$$

This relation was originally derived through empirical observations. When Fourier's law is incorporated into the first law of thermodynamics, a parabolic temperature field equation is produced. Fourier's law represents one of the best models in mathematical physics although it possesses anomalies, the most predominant being its prediction of heat flow at an infinite speed of propagation. This law noticeably breaks down in situations involving very short times, high heat fluxes, and at cryogenic temperatures where diffusion theory cannot account for short time inertial effects. However, Fourier's law is accurate and appropriate in describing heat conduction in most common engineering situations.

It is apparent in situations involving short times and cryogenic temperature that a more accurate constitutive law describing the nature of heat conduction be introduced. In order to associate a finite heat propagation velocity, while remaining within the continuum assumption, Vernotte [1] and others heuristically proposed the modified Fourier's law

$$\tau\frac{\partial q}{\partial t} + q = -k\frac{\partial T}{\partial x} \tag{2}$$

where τ is a finite thermal relaxation time. Physically, τ represents the finite buildup time for the commencement of heat flow in a medium. When the modified Fourier's law is used in conjunction with the expression for the conservation of energy, a hyperbolic temperature field equation [2] is attained. In this description of heat conduction, the thermal propagation speed c becomes finite for $\tau > 0$, namely $c = \sqrt{(\alpha/\tau)}$, where α is the thermal diffusivity. As $\tau \to 0$, the thermal propagation speed approaches infinity and the classical parabolic law is recovered. The modified Fourier's law has generally been accepted [3] as the next order approximation to the true nature of heat conduction. It appears that Fourier's law represents the lowest order approximation in the description of heat conduction.

In this exposition, we shall study the effects and ramifications of linear hyperbolic heat conduction in a composite medium. Composites are generally of great interest to engineers and physicists since they appear in numerous situations including: reinforced

NOMENCLATURE

a_1, a_{N+1} constants relating type of boundary condition
$a_j(\lambda_m, \xi)$ function associated with degenerate kernel
b_1, b_{N+1} constants relating type of boundary condition
$b_j(\lambda_m, \xi)$ function associated with degenerate kernel
B_{lm} set of constants defined by equation (23)
c_i propagation speed for region i
C_p specific heat
$c_j(\lambda_m, \xi)$ dependent variable defined by equation (29b)
k_i thermal conductivity for region i
$K(\xi, \xi_0; \lambda_m)$ degenerate kernel
$N(\lambda_m)$ normalization integral
$q_{w,j}(t)$ prescribed flux boundary function, $j = 1, N+1$
$q_i(x, t)$ heat flux for region i
$Q_i(\eta, \xi)$ dimensionless heat flux
$\bar{Q}_m(\xi)$ transform of dimensionless heat flux
$S(\eta, \xi)$ dimensionless heat source
t time variable
T_0 initial temperature
$T_{w,j}(t)$ prescribed temperature boundary functions, $j = 1, N+1$
$T_i(x, t)$ temperature for region i
$u_i(x, t)$ volumetric heat source in region i
x space variable.

Greek symbols
α_i thermal diffusivity for region i
η dimensionless space variable
$\theta_i(\eta, \xi)$ dimensionless temperature for region i
λ_m eigenvalues
ξ dimensionless time
ρ_i density for region i
τ_i relaxation time for region i
$\phi(\eta, \xi)$ auxiliary function
$\psi_{im}(\eta)$ eigenfunction for region i.

Superscripts
$*$ dimensionless quantity.

laminates, fins, reactor walls, stratified fluids, as well as many other applications. In hyperbolic heat conduction, the study of the interaction of dissimilar materials provides additional insight and understanding into the unusual behavior of heat conduction based on the modified Fourier's law. Recently, the special case of a two-region slab with a step change in exterior wall temperature was presented [4] using conduction based on the modified Fourier's law. This example displayed the unusual physics and some of the mathematical difficulties that are encountered in hyperbolic heat conduction for multilayered regions. However, no general formulations and subsequently no general analysis exist for hyperbolic heat conduction in composite media.

Presently, we develop the general one-dimensional temperature and flux formulations for the standard three orthogonal coordinate systems. Then, the general one-dimensional flux and temperature distributions are developed from the flux formulation using a generalized finite integral transform technique. This technique leads to an infinite set of coupled initial value problems in the transform variable. The set of ordinary differential equations is then transformed into an equivalent set of linear Volterra integral equations of the second kind with degenerate kernels. These integral equations are more amenable to numerical approximation than the original differential equations. The method of Bownds [10, 11] is incorporated to determine the transforms numerically. Once the flux distribution is ascertained, the temperature distribution is obtained from the conservation of energy. Finally, a numerical example considering a two-region composite with a pulsed volumetric source emanating from one region shall be investigated. In light of some recent experiments in superfluid helium [5, 6], in which a heat pulse was partially reflected at the interface of dissimilar materials, the results obtained here may lend some theoretical insight to heat conduction based on damped wave equations at cryogenic temperatures. Finally, the stark difference between parabolic and hyperbolic heat conduction shall be displayed.

GENERAL FORMULATION

Temperature and flux field equations

The governing one-dimensional temperature and heat flux field equations are now formulated for the three standard orthogonal coordinate systems. Referring to Fig. 1 and performing an energy balance [7] in the usual manner, one arrives at the following mathematical statement for the conservation of energy in any region i in the multiregion medium

$$\frac{-1}{x^p}\frac{\partial}{\partial x}(x^p q_i) + u_i(x, t) = (\rho C_p)_i \frac{\partial T_i}{\partial t},$$

$$x \in [x_i, x_{i+1}], \quad t > 0, \quad i = 1, 2, \ldots, N \quad (3)$$

where

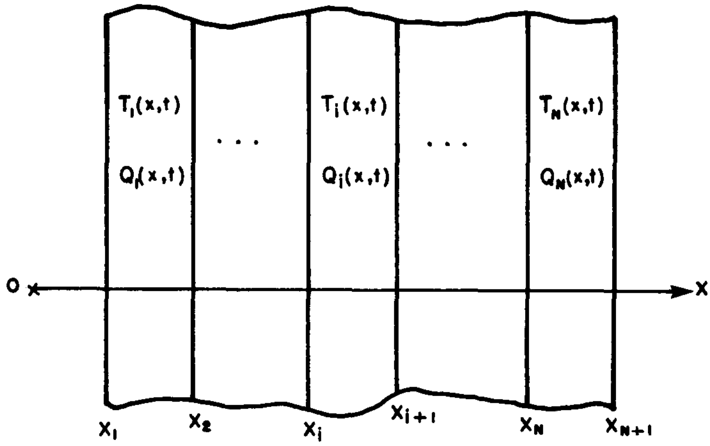

FIG. 1. Geometry and coordinates.

$$p = \begin{cases} 0 & \text{slab} \\ 1 & \text{cylinder} \\ 2 & \text{sphere.} \end{cases}$$

Eliminating $q_i(x,t)$ between equations (2) and (3) yields the following *temperature field equation* for any region i

$$\frac{1}{x^p}\frac{\partial}{\partial x}\left(x^p\frac{\partial T_i}{\partial x}\right)+\frac{1}{k_i}\left[u_i+\tau_i\frac{\partial u_i}{\partial t}\right] = \frac{1}{\alpha_i}\left[\tau_i\frac{\partial^2 T_i}{\partial t^2}+\frac{\partial T_i}{\partial t}\right],$$

$$x\in[x_i,x_{i+1}],\quad t>0,\quad i=1,2,\ldots,N. \quad (4)$$

Clearly as the relaxation time τ_i approaches zero, the standard parabolic case [7] is recovered.

The *heat flux field equation* is obtained by eliminating the temperature $T_i(x,t)$ between equations (2) and (3). We find

$$\frac{\partial}{\partial x}\left[\frac{1}{x^p}\frac{\partial}{\partial x}(x^p q_i)\right]-\frac{\partial u_i}{\partial x}=\frac{1}{\alpha_i}\left[\tau_i\frac{\partial^2 q_i}{\partial t^2}+\frac{\partial q_i}{\partial t}\right],$$

$$x\in[x_i,x_{i+1}],\quad t>0,\quad i=1,2,\ldots,N. \quad (5)$$

As seen in equations (4) and (5), the volumetric heat generation term $u_i(x,t)$ appears in different combinations in the two formulations. In some circumstances, it may be more convenient to choose one formulation over the other merely on the basis of this term. Another interesting feature of hyperbolic heat conduction in composites is that the associated homogeneous version of field equations (4) and (5), are not separable in the classical sense. For instance, if the product solution

$$q_i(x,t)=\psi_i(x)\Gamma(t) \quad (6)$$

is substituted into the homogeneous version of equation (5), separability is not achieved as in the parabolic composites [7]. The reason for the nonseparability is the presence of the coefficient τ_i as a multiplier to the second-order time derivative. Since τ_i assumes different values for different regions, a separated solution in the form given by equation (6) cannot be matched at the interfaces for all times. That is, the function $\Gamma(t)$ cannot be independent of all material properties as required by the classical eigenfunction expansion technique [7]. A generalized finite integral transform technique will be developed to overcome this difficulty.

Temperature formulation of boundary and initial conditions

First, we consider the appropriate boundary conditions associated with the temperature formulation. Convective heat transfer is not considered since hyperbolic heat conduction is valid for small time, i.e. times prior to the onset of mass movement. The mode of heat transfer between liquid–solid regions is considered only by conduction. For a specified temperature or heat flux at an external surface, the appropriate boundary condition becomes

$$-a_1k_1\frac{\partial T_1}{\partial x}+b_1T_1=a_1\left[\tau_1\frac{\mathrm{d}q_{w,1}}{\mathrm{d}t}+q_{w,1}(t)\right]+b_1T_{w,1}(t),\quad x=x_1 \quad (7a)$$

$$a_{N+1}k_N\frac{\partial T_N}{\partial x}+b_{N+1}T_N=a_{N+1}\left[\tau_N\frac{\mathrm{d}q_{w,N+1}}{\mathrm{d}t}+q_{w,N+1}(t)\right]+b_{N+1}T_{w,N+1}(t),\quad x=x_{N+1} \quad (7b)$$

where the general notation introduced above reduces to the prescribed temperature $T_{w,j}(t)$ or heat flux $q_{w,j}(t)$, $j=1,\ N+1$ boundary conditions depending on the various combination of the coefficients a_j and b_j. For example, if a prescribed temperature at $x=x_1$ were specified, then $a_1=0$, $b_1=1$, would be required. If however a specified heat flux is given, the coefficients would simply be $a_1=1$, $b_1=0$. A similar interpretation can be made when considering the boundary condition at $x=x_{N+1}$.

At the interface of two adjacent regions, two boundary conditions are required; namely, continuity of temperature, which implies perfect thermal contact, and continuity in heat flux. Continuity of temperature is expressed as

$$T_i = T_{i+1}, \quad x = x_{i+1} \quad i = 1,2,\ldots,N-1 \quad (7c)$$

while continuity in heat flux is

$$q_i = q_{i+1}, \quad x = x_{i+1} \quad i = 1,2,\ldots,N-1. \quad (7d)$$

Carefully eliminating both q_i and q_{i+1} from equation (7d), using equations (2) for each region i yields

$$-k_i\left[\tau_{i+1}\frac{\partial^2 T_i}{\partial x\,\partial t}+\frac{\partial T_i}{\partial x}\right] = -k_{i+1}\left[\tau_i\frac{\partial^2 T_{i+1}}{\partial x\,\partial t}+\frac{\partial T_{i+1}}{\partial x}\right], \quad x = x_{i+1}, \quad i = 1,2,\ldots,N-1 \quad (7e)$$

where $\tau_i \neq \tau_{i+1}$. If $\tau_i = \tau_{i+1}$, the usual parabolic heat conduction boundary condition is obtained

$$-k_i\frac{\partial T_i}{\partial x} = -k_{i+1}\frac{\partial T_{i+1}}{\partial x}, \quad x = x_{i+1}, \quad i = 1,2,\ldots,N-1, \quad t > 0. \quad (7f)$$

Equation (7e) represents a new linear, second-order boundary condition with mixed spatial and time derivatives which has not yet appeared in heat conduction. This new expression introduces additional complexity since it is nonseparable. As the relaxation times τ_i and τ_{i+1} approach zero, we see that equation (7e) is consistent with the parabolic formulation.

For convenience, the initial state of the medium is taken at the equilibrium temperature T_0. The initial conditions are readily established as

$$T_i = T_0, \quad \frac{\partial T_i}{\partial t} = 0, \quad t = 0, \quad x \in [x_i, x_{i+1}], \quad i = 1,2,\ldots,N. \quad (8a,b)$$

Once the temperature distribution has been determined, the heat flux distribution may be obtained by solving the modified Fourier's law, equation (2), for the heat flux distribution in each region i or by solving the energy conservation equation, equation (3), for $q_i(x,t)$.

Flux formulation of boundary and initial conditions

For a specified temperature or heat flux at an external surface, the appropriate boundary conditions in the heat flux variable are expressed as

$$a_1 q_1 - b_1\frac{1}{x^p}\frac{\partial}{\partial x}(x^p q_1) = a_1 q_{w,1}(t) + b_1\left[(\rho C_p)_1\frac{\mathrm{d}T_{w,1}}{\mathrm{d}t} - u_1\right], \quad x = x_1 \quad (9a)$$

and

$$a_{N+1} q_N - b_{N+1}\frac{1}{x^p}\frac{\partial}{\partial x}(x^p q_N) = -a_{N+1} q_{w,N+1}(t) + b_{N+1}\left[(\rho C_p)_N\frac{\mathrm{d}T_{w,N+1}}{\mathrm{d}t} - u_N\right], \quad x = x_{N+1} \quad (9b)$$

where the coefficients a_j, b_j, $j = 1$, $N+1$ have the same interpretation as the coefficients associated with temperature boundary conditions (7a) and (7b). In equations (9a) and (9b), conservation law (3) was used to convert a prescribed temperature boundary into the flux domain. If a prescribed surface temperature involves a step change in the surface temperature, then the terms involving the time derivative on the right-hand side of equations (9a) and (9b) will involve delta functions and the use of generalized functions will be necessary to preserve correctness in the formulation.

The statement of continuity of temperature at an interface can be formulated in the heat flux domain by taking the partial derivative with respect to time of equation (7c) and incorporating equation (3) to obtain

$$\frac{1}{(\rho C_p)_i}\left[-\frac{1}{x^p}\frac{\partial}{\partial x}(x^p q_i)+u_i\right] = \frac{1}{(\rho C_p)_{i+1}}\cdot\left[-\frac{1}{x^p}\frac{\partial}{\partial x}(x^p q_{i+1})+u_{i+1}\right], \quad x = x_{i+1}. \quad (9c)$$

Equation (9c) represents a separable but non-homogeneous boundary condition which is valid for both parabolic and hyperbolic heat conduction. Continuity of heat flux at the interfaces is simply expressed by equation (7d).

Considering the medium initially at the equilibrium state, the initial conditions are

$$q_i = 0, \quad \frac{\partial q_i}{\partial t} = 0, \quad t = 0, \quad i = 1,2,\ldots,N. \quad (10a,b)$$

Once the flux distribution has been established, the temperature distribution may be resolved by a time integration of the energy balance, equation (3), or by the spatial integration of the modified Fourier's law, equation (2), for each region i.

ANALYSIS

The previous formulation reveals that the field equations for temperature and heat flux, given by equations (4) and (5), respectively, have similar form except for the non-homogeneous source contribution. However, the statement of continuity of heat flux at the interfaces is quite convenient in the flux domain, as given by equation (7d), while the equivalent statement in the temperature variable, as given by equation (7e), represents a non-separable condition with mixed partial derivatives. Due to this added mathematical complexity, we choose to develop an analytical solution directly in the heat flux domain and then recover the temperature distribution by a time integration of the energy conservation law, equation (3). A generalized finite integral transform technique is developed, capable of yielding very accurate results for non-separable hyperbolic partial differential equations. This generalized transform technique reduces to the exact solution for situations where separability can be achieved, that is, when the relaxation times are identical in each region.

For convenience in the subsequent analysis, we introduce the following dimensionless quantities

$$\eta = \frac{c_1 x}{2\alpha_1}, \quad \xi = \frac{c_1^2 t}{2\alpha_1} \tag{11a,b}$$

where $c_1 = \sqrt{(\alpha_1/\tau_1)}$ and

$$Q_i(\eta,\xi) = \frac{q_i}{T_{\text{ref}}\left(\frac{k_1 c_1}{\alpha_1}\right)} \tag{11c}$$

$$\theta_i(\eta,\xi) = \frac{T_i - T_0}{T_{\text{ref}}} \tag{11d}$$

$$S_i(\eta,\xi) = \frac{u_i}{T_{\text{ref}}(k_1 c_1^2/4\alpha_1^2)} \tag{11e}$$

and the dimensionless property ratios

$$k_i^* = \frac{k_i}{k_1}, \quad \alpha_i^* = \frac{\alpha_i}{\alpha_1}, \quad c_i^* = \left(\frac{c_i}{c_1}\right)^2, \quad \tau_i^* = \frac{\alpha_i^*}{c_i^*},$$

$$i = 1,2,\ldots,N. \tag{12a–d}$$

The reference temperature T_{ref} is chosen to characterize the thermal disturbance of interest. Also, T_0 represents the initial temperature associated with the equilibrium initial state.

Introducing the dimensionless quantities expressed above into equations (5), (9) and (10) produces the general one-dimensional system governing the dimensionless flux distribution. The dimensionless flux field equation for a general one-dimensional composite is

$$L_{p,i}[Q_i(\eta,\xi)] = \frac{1}{2}\frac{\partial S_i}{\partial \eta} + \frac{2}{\alpha_i^*}\frac{\partial Q_i}{\partial \xi} \tag{13a}$$

where $L_{p,i}$ is the operator defined as

$$L_{p,i} \equiv \frac{\partial}{\partial \eta}\left[\frac{1}{\eta^p}\frac{\partial}{\partial \eta}(\eta^p)\right] - \frac{\tau_i^*}{\alpha_i^*}\frac{\partial^2}{\partial \xi^2} \tag{13b}$$

where p represents the particular geometry as expressed in equation (3) and i represents the region. The particular form of $L_{p,i}$ is naturally suggested by the dominant wave nature of the system under consideration. The dimensionless boundary conditions become

$$a_1^* Q_1 + b_1^*\left[-\frac{1}{\eta^p}\frac{\partial}{\partial \eta}(\eta^p Q_1)\right] = a_1^* Q_{w,1}(\xi)$$

$$+ b_1^*\left[-\frac{S_1}{2} + \frac{d\theta_{w,1}(\xi)}{d\xi}\right], \quad \eta = \eta_1 \tag{14a}$$

$$Q_i = Q_{i+1} \tag{14b}$$

$$\frac{\alpha_i^*}{k_i^*}\left[-\frac{1}{\eta^p}\frac{\partial}{\partial \eta}(\eta^p Q_i) + \frac{S_i}{2}\right]$$

$$= \frac{\alpha_{i+1}^*}{k_{i+1}^*}\left[-\frac{1}{\eta^p}\frac{\partial}{\partial \eta}(\eta^p Q_{i+1}) + \frac{S_{i+1}}{2}\right],$$

$$\eta = \eta_{i+1}, \quad i = 1,2,\ldots,N-1 \tag{14c}$$

$$-a_{N+1}^* Q_N + b_{N+1}^*\left[-\frac{1}{\eta^p}\frac{\partial}{\partial \eta}(\eta^p Q_N)\right]$$

$$= a_{N+1}^* Q_{w,N+1}(\xi) + b_{N+1}^*\left[-\frac{S_N}{2} + \frac{k_N^*}{\alpha_N^*}\frac{d\theta_{w,N+1}}{d\xi}\right],$$

$$\eta = \eta_{N+1}, \quad \xi > 0 \tag{14d}$$

where the boundary coefficients $a_j^*, b_j^*, j = 1, N+1$ are the dimensionless counterparts of the general notation introduced in equations (9a) and (9b). The dimensionless initial conditions become

$$Q_i = 0 \tag{15a}$$

$$\frac{\partial Q_i}{\partial \xi} = 0, \quad t = 0, \quad i = 1,2,\ldots,N. \tag{15b}$$

Equations (13)–(15) constitute the complete mathematical formulation to uniquely determine the dimensionless flux distribution once the thermal disturbances have been specified. Finally, the dimensionless equation governing the conservation of energy is

$$-\frac{1}{\eta^p}\frac{\partial}{\partial \eta}[\eta^p Q_i] + \frac{S_i(\eta,\xi)}{2} = \frac{k_i^*}{\alpha_i^*}\frac{\partial \theta_i}{\partial \xi},$$

$$\eta \in [\eta_i, \eta_{i+1}], \quad \xi \geq 0, \quad i = 1,2,\ldots,N. \tag{16}$$

We shall now develop a generalized finite integral transform technique capable of yielding accurate numerical results.

Generalized finite integral transform technique

Since equations (13)–(15) constitute a system which is nonseparable in the classical sense, as described with regard to equation (6), standard analytical solution techniques such as the finite integral transform fail. However, suppose we view the right-hand side of equation (13a) as an effective heat source or nonhomogeneity. We then develop the solution method by considering the auxiliary problem

$$L_{p,i}[\phi(\eta,\xi)] = 0, \quad \eta \in (\eta_i, \eta_{i+1}), \quad \xi > 0,$$

$$i = 1,2,\ldots,N \tag{17}$$

which is the classical (separable) wave equation.

The eigenfunctions $\psi_{im}(\eta)$ and eigenvalues λ_m are chosen by considering equation (17) and the homogeneous versions of equations (14a)–(14d), where $Q_i(\eta,\xi)$ is replaced by $\phi_i(\eta,\xi)$.

The appropriate eigenvalue problem then becomes

$$\frac{d}{d\eta}\left[\frac{1}{\eta^p}\frac{d}{d\eta}(\eta^p\psi_{im})\right]+\frac{\lambda_m^2\tau_i^*}{\alpha_i^*}\psi_{im}(\eta)=0 \quad (18a)$$

subject to the boundary conditions

$$a_1^*\psi_{1m}+b_1^*\left[-\frac{1}{\eta^p}\frac{d}{d\eta}(\eta^p\psi_{1m})\right]=0,\quad \eta=\eta_1 \quad (18b)$$

$$\frac{\alpha_i^*}{k_i^*}\left[-\frac{1}{\eta^p}\frac{d}{d\eta}(\eta^p\psi_{im})\right]=\frac{\alpha_{i+1}^*}{k_{i+1}^*}\left[-\frac{1}{\eta^p}\frac{d}{d\eta}(\eta^p\psi_{i+1,m})\right] \quad (18c)$$

$$\psi_{im}=\psi_{i+1,m},\quad \eta=\eta_{i+1},\quad i=1,2,\ldots,N-1 \quad (18d)$$

$$a_{N+1}^*\psi_{N,m}+b_{N+1}^*\left[-\frac{1}{\eta^p}\frac{d}{d\eta}(\eta^p\psi_{N,m})\right]=0,\quad \eta=\eta_{N+1} \quad (18e)$$

where $a_j^*, b_j^*, j=1, N+1$ are the boundary coefficients introduced in equation (14). The eigenvalue problem presented above is of the same type encountered in parabolic heat conduction [7].

The eigenfunctions obey the following important orthogonality relation

$$\sum_{i=1}^{N}\frac{\tau_i^*}{k_i^*}\int_{\eta=\eta_i}^{\eta_{i+1}}\eta^p\psi_{im}(\eta)\psi_{in}(\eta)\,d\eta=\begin{cases}0, & m\neq n\\ N(\lambda_m), & m=n\end{cases} \quad (19a)$$

where the normalization integral is defined as

$$N(\lambda_m)=\sum_{i=1}^{N}\frac{\tau_i^*}{k_i^*}\int_{\eta=\eta_i}^{\eta_{i+1}}\eta^p\psi_{im}^2(\eta)\,d\eta. \quad (19b)$$

Using this orthogonality relation, we now develop the integral transform pair as

inversion formula

$$Q_i(\eta,\xi)=\sum_{m}^{\infty}\frac{\psi_{im}(\eta)\bar{Q}_m(\xi)}{N(\lambda_m)} \quad (20a)\dagger$$

integral transform

$$\bar{Q}_m(\xi)=\sum_{i=1}^{N}\frac{\tau_i^*}{k_i^*}\int_{\eta=\eta_i}^{\eta_{i+1}}\eta^p\psi_{im}(\eta)Q_i(\eta,\xi)\,d\eta,\quad m=0,1,2,\ldots,\quad \xi\geqslant 0. \quad (20b)$$

Having defined the integral transform pair in equations (20a) and (20b), we remove all spatial dependence from equation (13a) by operating on it with

$$\frac{\alpha_i^*}{k_i^*}\int_{\eta=\eta_i}^{\eta_{i+1}}\eta^p\psi_{im}(\eta)\,d\eta \quad (21)$$

and sum over all regions $i=1,2,\ldots,N$. After some manipulation, the following ordinary differential equation for each m appears

$$\frac{d^2\bar{Q}_m(\xi)}{d\xi^2}+\lambda_m^2\bar{Q}_m(\xi)=A_m(\xi)-2\frac{d}{d\xi}\sum_{i=1}^{N}\frac{1}{k_i^*}\times\int_{\eta=\eta_i}^{\eta_{i+1}}Q_i(\eta,\xi)\eta^p\psi_{im}(\eta)\,d\eta \quad (22a)$$

where

$$A_m(\xi)=\sum_{i=1}^{N}\frac{\alpha_i^*}{k_i^*}\left[\psi_{im}\frac{\partial}{\partial\eta}(\eta^pQ_i)-Q_i\frac{d(\eta^p\psi_{im})}{d\eta}\right]_{\eta=\eta_i}^{\eta_{i+1}}-\frac{1}{2}\sum_{i=1}^{N}\frac{\alpha_i^*}{k_i^*}\int_{\eta=\eta_i}^{\eta_{i+1}}\frac{\partial S_i}{\partial\eta}\eta^p\psi_{im}(\eta)\,d\eta. \quad (22b)$$

An expression for $A_m(\xi)$ can be written in terms of the boundary conditions explicitly. Substituting the inversion formula, equation (20a), with the new dummy index l, into equation (22a) yields

$$\frac{d^2\bar{Q}_m(\xi)}{d\xi^2}+\lambda_m^2\bar{Q}(\xi)=A_m(\xi)-2\sum_{l}\frac{B_{lm}}{N(\lambda_l)}\cdot\frac{d\bar{Q}_l(\xi)}{d\xi} \quad (22c)$$

subject to the transformed initial conditions

$$\bar{Q}_m(0)=0 \quad (22d)$$

and

$$\frac{d\bar{Q}_m(0)}{d\xi}=0. \quad (22e)$$

The constants B_{lm} are defined as

$$B_{lm}=\sum_{i=1}^{N}\frac{1}{k_i^*}\int_{\eta=\eta_i}^{\eta_{i+1}}\eta^p\psi_{im}(\eta)\psi_{il}(\eta)\,d\eta. \quad (23)$$

Solution for transforms

The solution for the integral transform $\bar{Q}_m(\xi)$, as described in equation (22c), involves solving an infinite set of coupled ordinary differential equations with constant coefficients subject to the initial conditions expressed by equations (22d) and (22e). As shown in ref. [4], $A_m(\xi)$ may contain generalized functions, e.g. the Dirac delta function, which may cause some difficulty if equations (22c)–(22e) were solved directly by some numerical method.

Before proceeding with the general solution, we observe the similarity between the definition of B_{lm} expressed in equation (23) and the orthogonality relation expressed in equation (19a). For the situation in which all the regions in the composite have identical relaxation times, i.e. $\tau_i^*=1$ for all i, the coefficients defined in equation (23) reduce to

† If exterior BCs are of second kind, m starts at 0.

$$B_{lm} = \begin{cases} 0, & l \neq m \\ N(\lambda_m), & l = m. \end{cases} \quad (24)$$

For this case, equation (22c) for the transforms become uncoupled and can be solved directly to get

$$\bar{Q}_m(\xi) = \int_{\xi_0=0}^{\xi} \frac{A_m(\xi_0)}{\sqrt{(\lambda_m^2-1)}} e^{-(\xi-\xi_0)} \times \sin[\sqrt{(\lambda_m^2-1)}(\xi-\xi_0)]\, d\xi_0 \quad (25)$$

valid for $\tau_i^* = 1, i = 1, 2, \ldots, N$.

For the more general case of regions with different relaxation times, equations (22c)–(22e) must be solved simultaneously. Although the solution to these coupled equations may be formally expressed in an exact analytical form, the large number of terms typically required for hyperbolic systems makes this approach unfeasible when numerical results are required. As an alternative, we proposed in ref. [4] to convert equations (22c)–(22e) into an equivalent set of linear Volterra integral equations of the second kind. Using the Laplace transform technique and incorporating convolution, we arrive at

$$\bar{Q}_m(\xi) = f_m(\xi) - 2\sum_l \frac{B_{lm}}{N(\lambda_l)} \cdot \int_{\xi_0=0}^{\xi} \bar{Q}_l(\xi_0) \cos \lambda_m(\xi-\xi_0)\, d\xi \quad (26a)$$

where

$$f_m(\xi) = \int_{\xi_0=0}^{\xi} \frac{A_m(\xi_0)}{\lambda_m} \sin \lambda_m(\xi-\xi_0)\, d\xi_0. \quad (26b)$$

The 'kernel' present in equation (26a) namely $\cos \lambda_m(\xi, \xi_0)$ is degenerate [9], that is, it can be written as a finite sum of products of two linearly independent functions, one which depends on ξ and the other on ξ_0. We express this kernel as

$$K(\xi, \xi_0; \lambda_m) = \cos \lambda_m(\xi-\xi_0) = \sum_{j=1}^{2} a_j(\lambda_m, \xi) b_j(\lambda_m, \xi_0) \quad (27)$$

where the following form is chosen

$$a_1(\lambda_m, \xi) = \cos \lambda_m \xi, \quad a_2(\lambda_m, \xi) = \sin \lambda_m \xi,$$
$$b_1(\lambda_m, \xi_0) = \cos \lambda_m \xi_0, \quad b_2(\lambda_m, \xi_0) = \sin \lambda_m \xi_0. \quad (28a\text{–}d)$$

The method of Bownds [10–12] shall now be utilized in evaluating the integral equation representation of $\bar{Q}_m(\xi)$ as described by equation (26a). Briefly, this method transforms a general decomposable (degenerate) Volterra integral equation into a first-order initial value problem in a new variable. This technique is reminiscent of the usual solution technique applied to degenerate kernels in linear Fredholm theory. The method of Bownds may also be applied directly to semidegenerate integral equations. If a kernel is not exactly decomposable, a kernel approximation can be made to form a degenerate kernel. The numerical solution of this new smoother variable is obtained by a Runge–Kutta scheme, then the original transform variable $\bar{Q}_m(\xi)$ is reconstructed. Runge–Kutta and other initial value schemes are a paradigm to numerical analysis and many individual schemes are available with unique features. Fixed step size explicit integrators like the standard fourth-order Runge–Kutta or Butcher's method may be directly applied. Various explicit, semi explicit, and implicit Runge–Kutta methods [13] are available containing characteristics which may be exploited.

Writing equation (26a) in terms of the finite sum expressed in equation (27), yields

$$\bar{Q}_m(\xi) = f_m(\xi) - \sum_{j=1}^{2} a_j(\lambda_m, \xi) c_j(\lambda_m, \xi) \quad (29a)$$

where

$$c_j(\lambda_m, \xi) = 2\sum_l \frac{B_{lm}}{N(\lambda_l)} \cdot \int_{\xi_0=0}^{\xi} b_j(\lambda_m, \xi_0) \bar{Q}_l(\xi_0)\, d\xi_0, \quad j = 1, 2, \quad \xi \geqslant 0. \quad (29b)$$

Once $c_j(\lambda_m, \xi), j = 1, 2$ is determined numerically, the transform $\bar{Q}_m(\xi)$ is found from equation (29a).

The equations governing the new functions $c_j(\lambda_m, \xi)$ are obtained by differentiating equation (29b) and substituting equation (29a) into the result to get

$$\frac{dc_j(\lambda_m, \xi)}{d\xi} = 2\sum_l \frac{B_{lm}}{N(\lambda_l)} \cdot b_j(\lambda_m \xi) \cdot \left[f_l(\xi) - \sum_{k=1}^{2} a_k(\lambda_l, \xi) c_k(\lambda_l, \xi) \right], \quad j = 1, 2, \quad \xi \geqslant 0. \quad (30a)$$

The required initial conditions are obtained by evaluating equation (29b) at $\xi = 0$ to get

$$c_j(\lambda_m, 0) = 0, \quad j = 1, 2. \quad (30b)$$

We solve this set of equations, after truncating the l series after a finite number of terms, by a Runge–Kutta method. Once the c_j's are known, the integral transforms $\bar{Q}_m(\xi)$ are found from equation (29a) for each m. Finally, the flux distribution is resolved using the inversion formula expressed by equation (20a). If $\tau_i^* = 1$ for all regions, the exact analytical solution is obtained by substituting equation (25) into the inversion formula expressed by equation (20a) to get

$$Q_i(\eta, \xi) = \sum_m \frac{\psi_{im}(\eta)}{N(\lambda_m)} \int_{\xi_0=0}^{\xi} \frac{A_m(\xi_0)}{\sqrt{(\lambda_m^2-1)}} \cdot e^{-(\xi-\xi_0)} \sin[\sqrt{(\lambda_m^2-1)}(\xi-\xi_0)]\, d\xi_0, \quad i = 1, 2, \ldots, N, \quad \xi \geqslant 0. \quad (31)$$

In ref. [4], we obtained the temperature distribution by a spatial integration of the modified Fourier's law. However, in general one must use the conservation of energy equation as expressed in equation (16). A time integration of equation (16) yields

$$\theta_i(\eta,\xi) = \theta_i(\eta,0) - \frac{\alpha_i^*}{k_i^*}\int_{\xi_0=0}^{\xi}\frac{1}{\eta^P}\frac{\partial}{\partial\eta}[\eta^P Q_i(\eta,\xi_0)]\,d\xi_0 + \frac{\alpha_i^*}{k_i^*}\int_{\xi_0=0}^{\xi}\frac{S_i(\eta,\xi_0)\,d\xi_0}{2},$$

$$\xi \geq 0, \quad l = 1,2,\ldots,N \quad (32)$$

where $\theta_i(\eta,0) = 0$, $i = 1,2,\ldots,N$, from consideration of the equilibrium initial state.

In general, we substitute the inversion formula into equation (32) to get

$$\theta_i(\eta,\xi) = \frac{\alpha_i^*}{k_i^*}\left[\frac{-1}{\eta^P}\cdot\sum_m \frac{1}{N(\lambda_m)}\frac{d}{d\eta}(\eta\psi_{im}) \times \int_{\xi_0=0}^{\xi}\bar{Q}_m(\xi_0)\,d\xi_0 + \int_{\xi_0=0}^{\xi}\frac{S_i(\eta,\xi_0)}{2}\,d\xi_0\right],$$

$$i = 1,2,\ldots,N. \quad (33)$$

For the general case of regions with different relaxation times ($\tau_i^* \neq 1$ for at least one i), a relation for the integral of the transform can be developed using known functions and the numerically resolved c_j's. After some manipulation, we find

$$\int_{\xi_0=0}^{\xi}\bar{Q}_m(\xi_0)\,d\xi_0 = \int_{\xi_0=0}^{\xi} f_m(\xi_0)\,d\xi_0 + \sum_{j=1}^{2}\frac{c_j(\lambda_m,\xi)}{\lambda_m^2}\frac{da_j(\lambda_m,\xi)}{d\xi}. \quad (34a)$$

If $\tau_i^* = 1$ for all i, an exact evaluation of the integral of the transform leads to

$$\int_{\xi_0=0}^{\xi}\bar{Q}_m(\xi_0)\,d\xi_0 = -\frac{1}{\lambda_m^2}\int_{\xi_0=0}^{\xi}A_m(\xi_0) \times\left\{e^{-(\xi-\xi_0)}\left[\frac{\sin\sqrt{(\lambda_m^2-1)}(\xi-\xi_0)}{\sqrt{(\lambda_m^2-1)}} + \cos\sqrt{(\lambda_m^2-1)}(\xi-\xi_0)\right]-1\right\}d\xi_0$$

$$\tau_i^* = \tau_{i+1}^*, \quad i = 1,2,\ldots,N. \quad (34b)$$

Therefore, the temperature distribution in each region is known from equations (33) and (34a) for the general case and is known from equations (33) and (34b) when $\tau_i^* = 1$ for all i. If $\lambda_0 = 0$ ($m = 0$) is an allowable eigenvalue in the flux formulation, care must be taken in evaluating $A_0(\xi), f_0(\xi)$, and the integral of the transform. For $\lambda_m < 1$, expressions (25), (31) and (34b) containing $\sqrt{(\lambda_m^2-1)}$ are still real valued since the trigonometric functions of complex arguments become hyperbolic functions of real arguments.

The flux and temperature distributions are now completely available for the three standard orthogonal coordinate systems. When $\tau_i^* = 1$ for all i, an exact analytic solution was developed and will serve to check the numerical scheme. Next we will study a particular example considering a two region slab subjected to a pulsed source. The corresponding parabolic solution is also displayed to show the distinct differences in the two heat conduction approximations.

RESULTS

Numerical results displaying the unusual nature of hyperbolic heat conduction in a two region ($N = 2$) slab ($P = 0$) are presented. The fundamental nature of hyperbolic heat conduction is best represented by considering a pulsed volumetric source of width Δx emanating from region 1 adjacent to the exterior surface at $x = 0$. This source is described mathematically as

$$u_1(x,t) = \begin{cases} \dfrac{U_0\delta(t)}{\Delta x}, & x_1 = 0 < x \leq \Delta x \\ 0, & \Delta x < x \leq x_2 \end{cases}$$

$$u_2(x,t) = 0, \qquad x_2 < x < x_3 \quad (35a)$$

where

$$U_0 = \int_{t=0}^{\infty}\int_{x=0}^{\Delta x} u_1(x,t)\,dx\,dt < \infty. \quad (35b)$$

The term U_0 represents the total amount of energy generated by the source per area perpendicular to the x-direction for all space and time. By choosing the reference temperature as $T_{ref} = U_0 c_1/k_1$, we obtain the dimensionless sources

$$S_1(\eta,\xi) = \begin{cases} \dfrac{\delta(\xi)}{\Delta\eta}, & \eta_1 = 0 < \eta \leq \Delta x \\ 0, & \Delta\eta < \eta \leq \eta_2 \end{cases}$$

$$S_2(\eta,\xi) = 0, \qquad \eta_2 < \eta < \eta_3. \quad (35c)$$

We consider the situation where the two external boundaries are insulated for all time $t > 0$. In this situation, the total energy content of the system is preserved. The corresponding boundary coefficients are $a_1 = a_2 = 1$, $b_1 = b_2 = 0$. While the dimensionless counterparts become $a_1^* = a_2^* = 1$ and $b_1^* = b_2^* = 0$. Now, all the necessary ingredients for determining a unique solution are present. The parameters k_2^*, α_2^* and τ_2^* are varied in order to study the consequences and features associated with heat conduction based on the modified Fourier's law. A standard fourth-order Runge–Kutta [13] was considered in resolving $c_j(\lambda_m,\xi)$, $j = 1,2$, $m = 1,2,\ldots$ numerically when $\tau_2^* \neq \tau_1^*$. This explicit integrator was chosen for its combination of accuracy and ease of programming. When $\tau_1^* = \tau_2^*$, the exact analytic solution (both flux and temperature formulation) are presented. Comparison of the two formulations shows that the infinite series representing the flux and temperature distributions converge at different rates. The two formulations produce identical results in the limit, thus showing their equivalence and the correctness of the flux formulation. As is generally known, infinite series solutions for hyperbolic (wave) equations converge much slower than parabolic equations. The rate of convergence is related to the behavior of the terms in the bilinear series.

Figure 2 displays the flux distributions for the case where $k_2^* = 2$, $\tau_2^* = \alpha_2^* = 1$ for various times ξ as predicted by the hyperbolic and parabolic approximations of heat conduction. The dimensionless pulse width of the source is initially $\Delta\eta = 0.05$ and is initially released in the interval $0 \leqslant \eta \leqslant \Delta\eta$. At $\xi = 0.1$, the definite wave nature associated with hyperbolic heat conduction is evident. The initial pulse width doubles after being released at $t = 0^+$ since it initially has no preferred direction. Ahead of the wave lies an undisturbed region. This result is identical to the half space problem. In contrast, the parabolic approximation shows a continuous distribution which is smaller in magnitude. By $\xi = 0.7$, the initial wavefront has encountered the interface and has split into two waves traveling in opposing directions. One represents a reflected wave (traveling left) and the other represents the transmitted wave (traveling right) while both retain the initial wavelike features. This division initially occurred at $\xi = 0.45$ due to the impacting of the original wave into the interface at $\eta = 0.5$. In contrast, the parabolic approximation shows a monotonic decay toward steady state.

By $\xi = 1.3$, a pure reflection at the external boundaries has taken place. A pure reflection occurs because the insulated boundary conditions do not permit energy to leave the system. Each wavefront then approaches the interior interface at $\eta = 0.5$ at the same speed and will initially impact at $\xi = 1.45$. This impact appears to cancel the weaker of the two fronts as seen at $\xi = 1.9$. Meanwhile, parabolic heat conduction approximation has attained its steady state value.

Figure 3 displays the temperature distributions corresponding to the situation described in Fig. 2. Hyperbolic heat conduction predicts higher temperatures and fluxes than the parabolic approximation. The wavefront, of width $2\Delta\eta$, is preserved for all reflection–transmission effects when the wave speed in region 2 is unity, i.e. $c_2^* = c_1^* = 1$. At $\xi = 0.1$, the wave train shows the effects of diffusion in two ways with a residual temperature in the wake of the pulse and with a slant across the top of the initially flat wavefront. At $\xi = 0.7$, we see two distinct waves which result from the impacting of the initial wave at $\xi = 0.45$ into the interface at $\eta = 0.5$. Notice the wavefront traveling to the left is negative corresponding to temperatures below the initial temperature. Since $k_2^* > k_1^*$, less resistance causes more energy to enter region 2, thus creating the temperature lull of negative temperature wavefront in order to preserve the energy content of the system. Since the total energy is the spatial integral of the temperature times the heat capacity ρC_p, the energy content represented by each curve is identical since $(\rho C_p)_2$ changes in direct proportion to k_2^* when α_2^* is held constant. Also, since the external boundaries are insulated for all time $\xi > 0$, the total energy content is constant. At $\xi = 1.3$, the wave fronts have reflected from the exterior insulated surfaces and are directed toward the interface at $\eta = 0.5$. The dominant wave emanating from region 2 later combines with the subdominant wave to form a single wave moving toward the origin as seen clearly at $\xi = 1.9$. This transmission–reflection–combination phenomenon will persist until diffusion dominates.

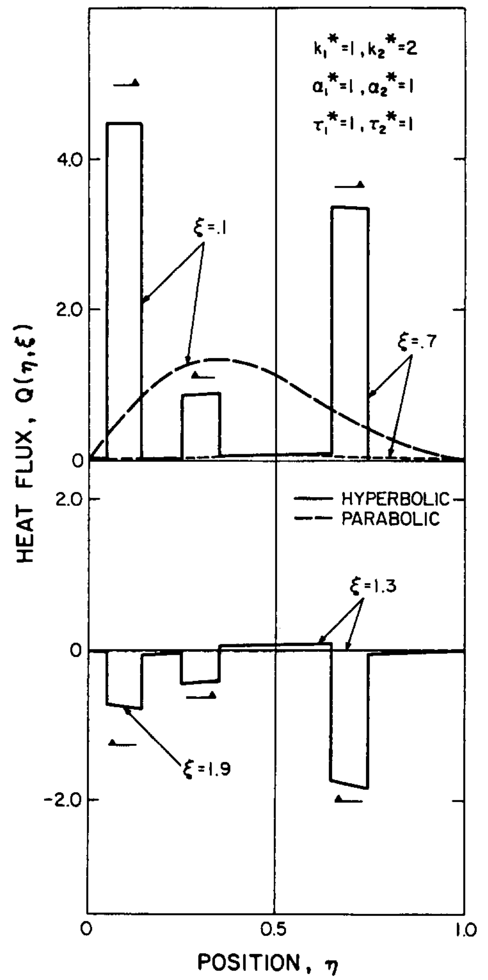

FIG. 2. Comparison of hyperbolic and parabolic heat flux distributions for a sequence of times with $k_2^* = 2$.

The remaining figures help provide a fundamental understanding into the effects of the parameters k_2^*, α_2^* and τ_2^* on hyperbolic heat conduction in composites. Figure 4 shows the effect of k_2^* on the temperature and flux distribution. The single region ($k_2^* = 1$) solution is presented as a reference to judge the effect of a higher and lower conductivity in region 2. At $\xi = 0.1$, the temperature distributions for all three cases considered are identical since the wavefront is unaware of region 2 at this time. At $\xi = 0.7$, the distinction between the three values of k_2^* becomes clear.

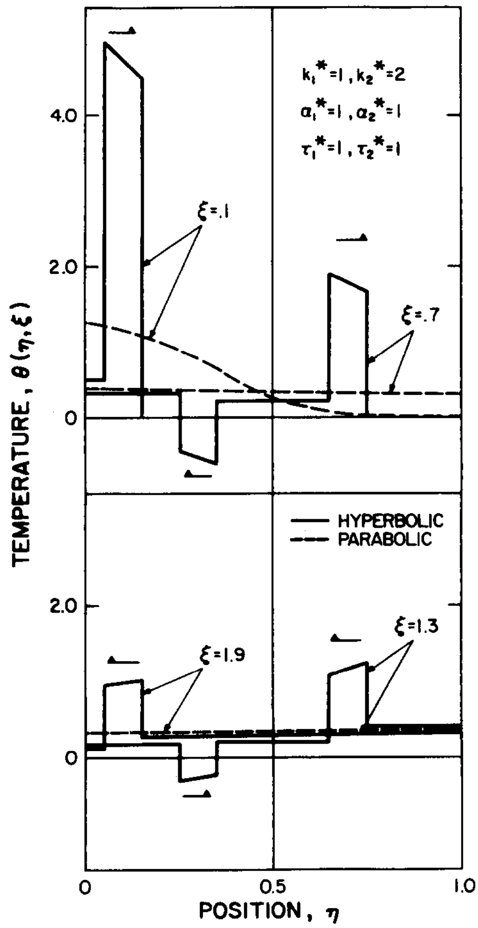

FIG. 3. Comparison of hyperbolic and parabolic temperature distributions for a sequence of times with $k_2^* = 2$.

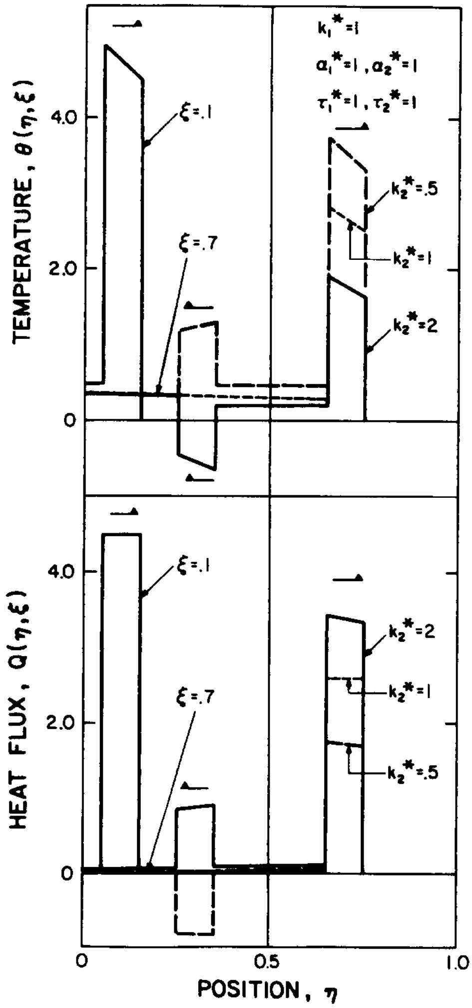

FIG. 4. Effect of k_2^* on temperature and heat flux.

When $k_2^* = 2 > k_1^*$, the forward wavefront is smaller in magnitude compared to the equivalent one-region standard. The reflected portion, which initially encountered the interface at $\xi = 0.45$, shows a wave moving toward the origin but negative in magnitude. Again, this temperature lull is due to the enhanced ability of region 2 to transmit energy and the basic criteria of energy conservation. When $k_2^* = 0.5$, the forward temperature wave is greater than the one-region ($k_2^* = 1$) counterpart, while a positive magnitude wave is reflected toward the origin. Again, this is consistent with energy considerations. The corresponding flux distributions display similar features.

The effect of various diffusivities in region 2 are displayed in Fig. 5. Since the wave speed ratio is $\sqrt{c_2^*} = \sqrt{(\alpha_2^*/\tau_2^*)}$, we expect different speeds in region 2 when $\alpha_2^* \neq 1$. At $\xi = 0.1$, all curves are again identical. At $\xi = 0.7$, the distinctions between the various diffusivities $\alpha_2^* = 0.5$, 1, 2 are evident. When $\alpha_2^* = 2$, the wave speed c_2^* is greater than the one-region ($c_2^* = 1$) standard, giving rise to a stretching of the width of the pulse disturbance. At $\xi = 0.45$, the leading edge of the pulse impacts the interface at $\eta = 0.5$, and the wave emerging from the interface in region 2 is pulling away faster ($c_2^* > c_1^*$) than the oncoming energy from region 1. Hence, the net effect is the stretching of the pulse width. This is evident in both the temperature and flux distributions. The opposite occurs when $c_2^* < c_1^*$, as seen when $\alpha_1^* = 0.5$. As the energy leaves the interface when $\xi = 0.45$, a 'pile-up' of energy occurs since the wave leaving the interface moves slower than that entering the interface, hence a thinner pulse width emerges. The reflected portion, when $\alpha_2^* = 2$, shows a temperature wave directed toward the origin consistent with the energy content of the system. Also, the reflected wave corresponding to $\alpha_2^* = 0.5$, displays a negative temperature wave directed toward the origin. This again is consistent with conservation of energy in the system.

Finally, the effect of various relaxation times in region 2 is studied. For fixed $k_2^* = \alpha_2^* = 1$, the wave speed in region 2 should change for various relaxation times τ_2^*. In Fig. 6, the distributions plotted are obtained from the numerical procedure outlined in the analysis section since no exact solution is available. When $\xi = 0.7$, three distinct curves exist for the various relaxation times in region 2 for $\tau_2^* = 0.5$, 1, 2. When $\tau_2^* = 2$, the wave speed in region 2 is less than

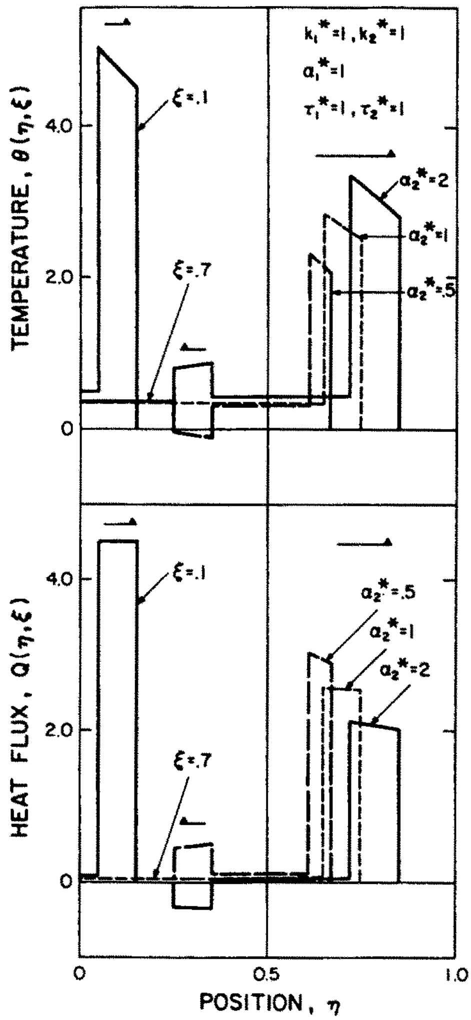

FIG. 5. Effect of α_2^* on temperature and heat flux.

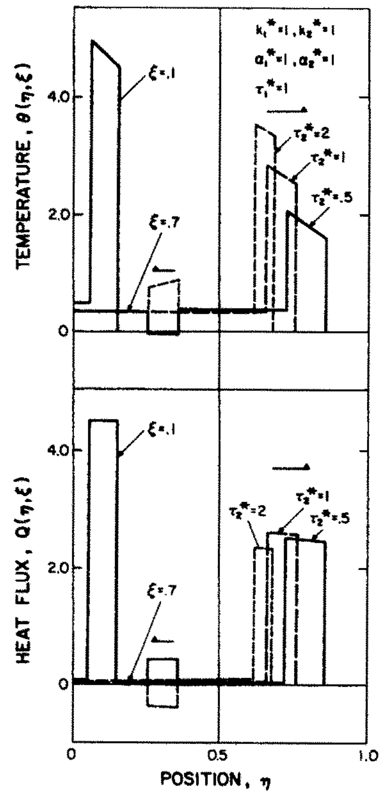

FIG. 6. Effect of τ_2^* on temperature and heat flux.

the one-region standard ($\tau_2^* = 1$). Again, a compressed pulse appears due to the dissimilar wave speeds of the two regions. The wave speed increases when $\tau_2^* = 0.5$ and the pulse width expands accordingly. The total energy content of the system is preserved for each temperature distribution.

CONCLUSIONS

The general constant property heat flux and temperature formulation, based on the hyperbolic heat conduction approximation, has been developed for multiregion media. These new formulations produce a linear but non-separable system. In addition, the temperature formulation produces a new non-separable higher order boundary condition at the interface of two adjacent regions. These formulations recover the standard parabolic approximation as the relaxation times in each region approach zero. Since these systems of equations are nonseparable, a generalized finite integral transform technique is proposed which produces an infinite set of coupled, non-homogeneous, constant coefficient ordinary differential equations in the transformed variable. Since these equations may contain generalized functions and since a large number of terms of the infinite series are required, standard numerical solution techniques are not well suited. Thus we transform these ordinary differential equations into an infinite set of linear Volterra integral equations of the second kind. Since the kernels are exactly decomposable, the method of Bownds is applied in a unique and non-standard manner to numerically solve for the transforms. An exact analytical solution for the temperature and flux distributions is available when the relaxation times for each region are identical. The accuracy of the proposed flux formulation and numerical procedure may then be tested. The analysis presented here has the potential application to non-linear problems.

The contrast between the parabolic and hyperbolic heat conduction approximations in a two-region slab exposed to a pulsed volumetric source is displayed. The numerical procedure for finding the integral transform is shown to be numerically accurate for the test cases where the relaxation times are identical in both regions. The figures displaying both the exact analytical and numerically accurate profiles are identical. It is shown that internal reflections at the interface of two dissimilar media are present and the extent of the reflection varies with properties. For various combinations of material properties, the wave may change speeds and deformation of the pulse width may occur. In light of some recent experiments involving reflections of thermal waves in superfluid helium [5, 6] the present investigation will help to provide a sound theory for predicting this complex phenomenon.

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REFERENCES

1. M. P. Vernotte, Les Paradoxes de la théorie continue de l'equation de la chaleur, *Comptes Rendus* **246**, 3154–3155 (1958).
2. B. Vick and M. N. Özisik, Growth and decay of a thermal pulse predicted by the hyperbolic heat conduction equation, *J. Heat Transfer* **105**, 902–907 (1983).
3. S. Simons, On the differential equation for heat conduction, *Transp. Theory Statist. Phys.* **2**, 117–128 (1972).
4. J. Frankel, B. Vick and M. N. Özisik, Hyperbolic heat conduction in composite regions, *Eighth International Heat Transfer Conference*, San Francisco, California, 17–22 August (1986).
5. J. R. Torczynski, On the interaction of second shock waves and vorticity in superfluid helium, *Physics Fluids* **27**(11), 2636–2644 (1984).
6. J. R. Torczynski, D. Gerthsen and T. Roesgen, Schlieren photography of second-sound shock waves in superfluid helium, *Physics Fluids* **27**(10), 2418–2423 (1984).
7. M. N. Özisik, *Heat Conduction*. Wiley, New York (1980).
8. D. Rankrishna and N. R. Amundson, Transport in composite materials: reduction to self adjoint formalism, *Chem. Engng Sci.* **29**, 1457–1464 (1974).
9. C. D. Green, *Integral Equations Methods*. Barnes and Noble, New York (1969).
10. J. M. Bownds and B. Wood, On numerically solving nonlinear Volterra integral equations with fewer computations, *SIAM J. Numer. Anal.* **13**(5), 705–719 (1976).
11. J. M. Bownds, A modified Galerkin approximation method for Volterra equations with smooth kernels, *Appl. Math. Computations* **4**, 67–79 (1978).
12. M. N. Golberg (Editor), *Solution Methods for Integral Equations*. Plenum Press, New York (1978).
13. J. D. Lambert, *Computational Methods in Ordinary Differential Equations*. Wiley, New York (1977).
14. P. W. Berg and J. L. McGregor, *Elementary Partial Differential Equations*, p. 366. Holden-Day, San Francisco (1966).

FORMULATION GENERALE ET ANALYSE DE LA CONDUCTION HYPERBOLIQUE DE LA CHALEUR DANS LES MILIEUX COMPOSITES

Résumé—Quelques résultats expérimentaux récents montrent l'existence de réflexion d'ondes thermiques à l'interface de matériaux différents dans l'hélium superfluide. A la lumière de ces résultats, une recherche sur les ondes thermiques dans les composites est faite pour fournir une base théorique au phénomène observé. On présente une formulation générale de la conduction monodimensionnelle hyperbolique dans un milieu composite. La solution générale, basée sur la formulation du flux, est développée pour les systèmes à trois dimensions orthogonales. Contrairement à la conduction parabolique classique, la conduction basée sur la loi modifiée de Fourier produit des équations de champ inséparables pour, à la fois, la température et le flux et par suite les techniques analytiques classiques ne peuvent être appliquées. De façon à surmonter cette difficulté, on propose une technique de transformation intégrale, finie, générale pour le domaine du flux et une solution générale est développée pour les systèmes tridimensionels. La solution générale est appliquée au cas d'une plaque à deux régions, avec une source volumique pulsée et des surfaces externes isolées, qui montre la nature, inhabituelle et cause de controverses, associée à la conduction basée sur la loi de Fourier modifiée dans les régions composites.

ALLGEMEINER ANSATZ UND ANALYSE DER HYPERBOLISCHEN WÄRMELEITUNG IN VERBUNDMATERIALIEN

Zusammenfassung—Einige neue experimentelle Ergebnisse zeigen die Existenz der Reflektion von thermischen Wellen an der Grenzfläche von verschiedenartigen Materialien in supraleitendem Helium. Angesichts dieser Ergebnisse wird eine theoretische Untersuchung von thermischen Wellen vorgestellt, die eine theoretische Grundlage des beobachteten Phänomens liefert. Ein allgemeiner eindimensionaler Ansatz für die Temperatur und den Wärmestrom bei hyperbolischer Wärmeleitung in Verbundmaterialien wird vorgestellt. Die allgemeine Lösung, die auf dem Ansatz für den Wärmestrom basiert, wird für die üblichen drei orthogonalen Koordinatensysteme entwickelt. Abweichend von der klassischen parabolischen Wärmeleitung, ergibt die auf dem modifizierten Fourier-Gesetz basierende Wärmeleitung nichtlösbare Feldgleichungen für die Temperatur und den Wärmestrom, weshalb analytische Standardtechniken nicht angewendet werden können. Um diese Schwierigkeiten zu erleichtern, wird für den Wärmestrom ein generalisiertes Endlich-Integral-Transformationsverfahren vorgeschlagen und eine allgemeine Lösung für die üblichen drei orthogonalen Koordinatensysteme entwickelt. Die allgemeine Lösung wird auf den Fall einer Zwei-Schichten-Platte mit einer gepulsten Volumenquelle und isolierten Außenflächen angewendet, die die ungewöhnliche und kontroverse Beschaffenheit der auf dem modifizierten Fourier-Gesetz basierenden Wärmeleitung in zusammengesetzten Bereichen darstellt.

ОБЩАЯ ФОРМУЛИРОВКА И АНАЛИЗ ГИПЕРБОЛИЧЕСКОЙ ЗАДАЧИ ТЕПЛОПРОВОДНОСТИ В СЛОЖНЫХ СРЕДАХ

Аннотация—Некоторые последние экспериментальные результаты показывают, что в сверхтекучем гелии наблюдается отражение тепловых волн на границе разнородных материалов. В свете этих результатов предпринято теоретическое исследование тепловых волн в сложном материале для теоретического обоснования наблюдаемого явления. Разработана общая постановка одномерной гиперболической задачи теплопроводности в сложной среде для температуры и теплового потока. Общее решение, основанное на формулировке задачи для теплового потока, представлено в стандартных ортогональных системах координат. В отличие от классической параболической задачи теплопроводности модифицированный закон Фурье дает несепарабельные уравнения для полей температуры и теплового потока, и поэтому стандартные аналитические методы не применимы в данном случае. Для уменьшения этих трудностей обобщенный метод конечных интегральных преобразований преложен для уравнения теплового потока, а общее решение получено для стандартных ортогональных систем координат. Общее решение применяется для случая двухзонной пластины с пульсирующим объемным источником и изолированными внешними поверхностями, который необычен и сложен, что связано с использованием модифицированного закона Фурье для сложных материалов.